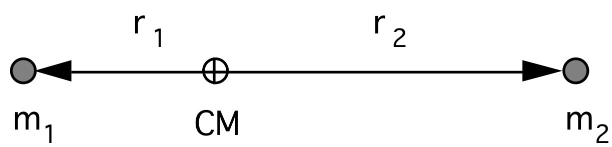
Two-Body Problem

All the derivations above were based on the idea that the sun did not move in any of this and made pretend it was only the planet that did the moving – even though Newton's Laws of Motion would dictate that the sun should feel the same forces and undergo accelerations also. If we are dealing with small planets going around massive stars, then the above are pretty reasonable approximations of what happens. However, if we are in fact dealing with big planets and small stars, or even two stars together, we need to generalize the above results. In physics, this is known as the "Two-Body Problem."

Imagine that we have two objects, m_1 and m_2 that are orbiting each other. (We will ignore the rest of the universe.) To make it easier, we will also put ourselves in the reference frame for which the center of mass is not moving. The picture below shows the two objects, and the position vectors, r_1 and r_2 that originate from the center of mass of the two objects.



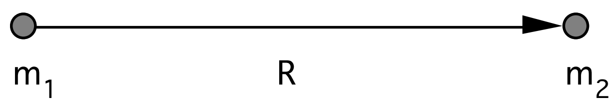
From the definition of center of mass, we can say that

$$m_1\vec{r}_1 + m_2\vec{r}_2 = 0$$

Which we can rewrite as

$$\vec{r}_1 = -\frac{m_2}{m_1}\vec{r}_2$$

We could also define the vector **R** to be the position of m_2 with respect to m_1, as shown below.



From the two diagrams, we can say

$$\vec{R} = \vec{r}_2 - \vec{r}_1$$

We can then substitute for r_1 to get

$$\vec{R} = \vec{r}_2\left(1 + \frac{m_2}{m_1}\right)$$

We can rewrite this as

$$\vec{r}_2 = \left(\frac{m_1}{m_1 + m_2}\right)\vec{R}$$

Putting our origin at the center of mass, since that does not move, we can then write Newton's Second Law for m_2 as

$$\sum F = m_2\vec{a}_2$$

For a general force, *f(R)*, that only depends on the distance *R* between the two masses we would therefore say

$$f(R)\hat{r} = m_2\vec{a}_2$$

That is the general central force problem that we have been solving for the past 8 pages! In addition, the acceleration vector $\mathbf{a}_2$ in the above is the second derivative of the position vector $\mathbf{r}_2$, so we can rewrite that using the results from above to say

$$f(R)\hat{r} = m_2\left(\frac{m_1}{m_1 + m_2}\right)\ddot{R}$$

If we define $\mu = (m_1m_2)/(m_1+m_2)$ then we can say

$$f(R)\hat{r} = \mu\ddot{R}$$

Two-Body Problem

The term μ is called the "reduced mass" because it has units of mass, and turns the original two-body problem into the one-body central force problem that we began with. Keep in mind that we could have done all the above, but reverse the directions and subscripts, and we would still end up with the above statement. Saying that *f(R)* is Newton's Universal Gravitation we could therefore say that

$$-G\frac{m_1 m_2}{R^2}\hat{r} = \mu\ddot{R}$$

This is basically where we started to show that the orbit is a conic section! That means that the result of this differential equation would also give us a conic section. The strange part of this is that the "R" in the equation above is the *relative* position of one mass with respect to the other. Therefor the relative position of one mass with respect to the other is an ellipse. Since there was no real difference in which mass was the reference frame, each mass think the other mass is orbiting the fir

The orbits of the two masses would still be ellipses (or conic sections) and the focus of each orbit will be the center of mass.